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OPTIMIZATION OF FUZZY INVENTORY MANAGEMENT IN INDUSTRIAL PROCESSES USING DEEP LEARNING ALGORITHMS: A HYBRID APPROACH FOR ENHANCING DEMAND FORECASTING AND SUPPLY CHAIN EFFICIENCY

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Abstract:
In today's dynamic business landscape, effective inventory management is essential for minimizing costs and maximizing profitability. Traditional models like EOQ and JIT often fall short in handling demand and supply uncertainties due to their reliance on precise data. This paper introduces a novel approach that combines fuzzy logic and deep learning to address these limitations. Fuzzy logic offers a robust framework for decision-making under uncertainty, while deep learning improves predictive accuracy by identifying complex patterns in historical data. By transforming data into fuzzy sets and applying neural networks for demand forecasting, the proposed model optimizes inventory levels to reduce costs and prevent stockouts. A mathematical model and algorithmic implementation demonstrate the approach's effectiveness and a numerical example highlights improvements in inventory control, including reduced holding costs. This study underscores the potential of integrating AI techniques for adaptive, data-driven inventory management with broad applications across various industrial processes.

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1. INTRODUCTION

Inventory management is a cornerstone of efficient supply chain operations, directly influencing operational performance and customer satisfaction. Properly managed inventory ensures that products are available to meet customer demand without excessive overstock or stockouts, balancing cost-efficiency and service levels. However, achieving this balance is challenging, especially in today's rapidly evolving and uncertain market environments. Traditional inventory management models, such as Economic Order Quantity (EOQ) and Just-In-Time (JIT), have been widely used to optimize inventory decisions, but they rely on precise demand forecasts, which are often difficult to obtain in real-world scenarios. This limitation becomes even more pronounced in volatile markets where demand and supply constantly fluctuate.

EOQ, for example, assumes that demand is known and constant, while JIT relies heavily on accurate predictions of customer needs and supplier reliability. Such models may work well in stable environments, but they must be equipped to handle the complexities of modern supply chains, which are increasingly characterized by unpredictability. Factors such as varying consumer behavior, global supply chain disruptions, and shifts in market trends introduce significant uncertainty, making traditional models less effective. As a result, businesses often struggle to

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manage inventory efficiently, leading to either excess stock, which ties up capital and increases holding costs, or insufficient stock, which results in lost sales and dissatisfied customers.

Fuzzy logic-based models have been developed to address the inherent uncertainties in inventory management. These models provide a flexible framework that allows decision-makers to incorporate imprecise or vague information, such as uncertain demand forecasts and variable lead times, into their inventory control strategies. Fuzzy inventory models replace the rigid assumptions of traditional models with a more adaptable approach that can better handle ambiguity and uncertainty. For instance, rather than assuming exact values for demand, fuzzy models use linguistic variables (e.g. “high demand”, “moderate supply”) and membership functions to represent uncertainty, making the model more aligned with real-world scenarios.

However, despite their ability to handle uncertainty, fuzzy logic models often lack the predictive power required for making highly accurate inventory decisions. While helpful in accommodating vagueness, these models are typically not designed to process large amounts of data or uncover complex patterns in demand and supply trends. This limitation hinders their ability to provide precise inventory recommendations, especially in Industrial Processes where data-driven decision-making is becoming increasingly critical.

Recent advancements in artificial intelligence (AI), particularly in deep learning, offer a promising solution to this challenge. Deep learning algorithms have revolutionized various fields, including image recognition, natural language processing, and even supply chain management, due to their ability to process and learn from vast amounts of data. In inventory management, deep learning models excel at demand forecasting, as they can identify intricate patterns and relationships within historical data that traditional models or human decision-makers might miss. This predictive capability makes deep learning an ideal complement to fuzzy logic, as it can enhance the accuracy of forecasts, providing more reliable inputs for inventory decisions.

This paper proposes an integrated approach that combines the strengths of both fuzzy logic and deep learning to optimize inventory management. Fuzzy logic’s ability to handle uncertainty and deep learning’s powerful data-driven prediction capabilities provide a robust framework for managing inventory in dynamic environments. Integrating these two methodologies allows businesses to navigate the complexities of modern supply chains more effectively by providing more flexible, adaptive, and accurate inventory solutions.

A novel inventory management model was introduced, integrating fuzzy logic with deep learning to enhance decision-making under uncertainty. A mathematical framework was developed for this integration, utilizing fuzzy sets to represent uncertainty and deep learning algorithms to predict demand and supply trends. Through numerical examples, the study demonstrated how this integrated approach outperformed traditional inventory models in managing real-world uncertainties. Potential applications of the model were explored across various industrial processes, and avenues for future research were identified. The sections that follow review the relevant literature, outline the methodology, present a detailed case study, and discuss the findings. This research contributes to the growing field of AI-driven inventory management and provides practical insights for businesses seeking to improve supply chain efficiency in an uncertain, data-rich environment.

2. LITERATURE REVIEW

In their study on perishable supply chains, Ali et al. [1] explore a multi-product, multi-echelon supply chain system under uncertain and dynamic conditions using a fuzzy non-linear programming approach. The authors emphasize the unique challenges of managing perishable goods, including the need to account for perishability in logistics planning. They introduce a fuzzy logic framework that optimizes inventory levels across various echelons, to minimize wastage while meeting customer demand. Their model underscores the importance of integrating fuzzy decision-making techniques in supply chains to address uncertainties inherent to perishable items.

Akram and Ullah [2] address Z-number-based multi-objective linear programming by incorporating different membership functions to represent uncertain and imprecise information. This study enhances traditional fuzzy models by adopting Z-numbers, allowing for a more nuanced representation of uncertainty in multi-objective problems. The paper highlights the relevance of Z-numbers in real-world applications where ambiguity in decision variables affects outcomes, such as supply chain management and economic planning.

Bahmani, et al. [3] integrate two-stage assembly flow shop scheduling with vehicle routing in logistics optimization, employing an enhanced Whale Optimization Algorithm (WOA). The study demonstrates how this algorithmic approach can streamline scheduling and routing tasks in industrial settings, leading to lower transportation costs and more efficient production schedules. Their approach highlights the utility of bio-inspired algorithms in handling complex, multi-stage problems within industrial engineering.

Cheng, et al. [4] focus on disaster waste cleanup using a multi-period, two-echelon location routing problem (LRP). The study develops an optimization model that prioritizes both efficiency and environmental considerations in waste management following natural disasters. This research contributes significantly to urban planning and disaster management by optimizing logistical and location-related decisions that mitigate environmental impact in post-disaster waste scenarios.

In their seminal work on economic order quantity (EOQ), Chand and Ward [5] examine the impacts of permissible payment delays on inventory decisions. This early research in operations management provides foundational insights into how payment terms can affect order quantity, particularly in scenarios where suppliers allow deferred payments. Their findings support the development of inventory models that incorporate financial conditions as part of EOQ calculations.

Ghosh, et al. [6] explore sustainable waste management solutions through a multi-objective solid transportation problem framed within a Pythagorean hesitant fuzzy environment. By addressing both carbon emissions and cost factors, the study demonstrates how fuzzy optimization techniques can support environmentally sustainable practices in waste logistics, emphasizing the need for flexible decision-making tools in the context of environmental constraints.

Ghodousian, et al. [7] propose a non-linear generalization for optimization problems constrained by continuous max-t-norm fuzzy relational inequalities. Their model extends traditional optimization approaches to account for complex, fuzzy relational structures, showing promise for applications in areas where conventional linear models may be too restrictive. This paper is particularly relevant for advancing fuzzy logic applications in optimization.

This review of multi-objective mixed-integer non-linear programming (MINLP) methods by Jaber, et al. [8] presents various approaches for handling complex decision-making problems with competing objectives. They evaluate the strengths and limitations of prominent MINLP techniques, providing a comprehensive resource for practitioners and researchers dealing with non-linear ities in optimization models across diverse sectors.

Jenifer [9] introduces a methodology for solving fuzzy linear programming (FLP) problems by treating them as fuzzy linear complementarity problems (FLCPs). This approach bridges FLP and FLCP, providing a new solution technique that expands the toolbox for fuzzy optimization. The study showcases the potential for addressing uncertainty in linear programming models, especially where traditional crisp solutions are inadequate.

Chung’s [10] study offers a theorem for determining EOQ in scenarios with permissible payment delays, providing a formalized solution approach for businesses facing such financial arrangements. His model enhances inventory management practices, particularly for firms where cash flow constraints influence ordering decisions.

Khalifa, et al. [11] explore the application of neutrosophic Kuhn-Tucker conditions for solving two-level linear programming problems under Pythagorean fuzzy conditions. Their study introduces a novel approach for handling uncertainty in hierarchical decision-making structures, proving relevant for complex multi-level organizational models.

This research focuses on fuzzy non-linear programming (FNLP) using Beale’s conditions with trapezoidal membership functions. Kaliyaperumal and Muralikrishna’s [12] model addresses optimization in settings where non-linearity and fuzziness overlap, making it applicable to industries where decision variables follow non-linear patterns influenced by uncertainty.

Pal, et al. [13] study decision-making in a dual-channel competitive green supply chain, factoring in promotional efforts as a variable. Their findings provide insights into how promotional activities impact supply chain efficiency, with implications for companies that aim to balance profitability with sustainability.

Considering stochastic deterioration, Pervin, et al. [14] examine an inventory control model under time-dependent demand and time-varying holding costs. Their model provides a practical solution for inventory systems where product longevity and demand fluctuate, a situation common in perishable goods.

Prasad, et al. [15] use a fuzzy goal programming approach to solve linear fractional programming problems, demonstrating an effective method for optimizing objectives in uncertain conditions. This approach holds significance for fields like finance and operations, where multiple, often competing, goals must be achieved under ambiguity.

Chaudhary, et al. [16] present a sustainable inventory model addressing defective items within a fuzzy framework. Their model aligns with sustainable practices by accounting for defects in inventory systems, making it applicable to industries focused on quality control and sustainability.

San-Jose, et al. [17], paper develops an optimal policy for inventory management where demand is influenced by price, time, and advertisement frequency. Their work has implications for retail and marketing, providing a model that aligns inventory levels with consumer demand dynamics.

Supian, et al. [18] propose a ride-hailing matching model using interval-valued fuzzy multi-objective linear programming, tackling uncertain travel times in ride-sharing. Their model supports the development of more reliable and efficient ride-hailing systems in urban environments.

Seikh and Dutta [19] introduce a non-linear approach to matrix games with picture fuzzy payoffs, applying it to cyberterrorism scenarios. Their methodology illustrates the utility of fuzzy models in security applications, where uncertainty and high stakes require robust decision-making.

Shivani, et al. [20] explore multi-objective non-linear programming for waste management, introducing rough interval parameters. This work aligns with sustainable waste management practices by providing a flexible model that accommodates the complexity of urban waste systems.

Yu, et al. [21] address a multi-objective location routing problem for waste collection using metaheuristics. Their model optimizes routing efficiency, demonstrating its utility for large-scale waste management operations that aim to minimize travel distances and operational costs.

Wang, et al. [22] examine a two-echelon multi-period location routing problem, sharing transportation resources across multiple tasks. Their model is valuable for optimizing logistical networks where resource-sharing can reduce overall operational costs and enhance efficiency.

2.1 Research Gap

The integration of real-time data streams into the model, such as dynamic pricing and customer feedback, remains an area that needs further exploration. Future research can focus on how the model can process and adapt to real-time information to make more responsive inventory decisions. While this paper focuses on inventory optimization, real-world businesses often need to consider multiple objectives (e.g., minimizing costs, maximizing customer satisfaction, and reducing carbon footprint). Research into multi-objective fuzzy-deep learning inventory models could provide more comprehensive solutions. More research is needed to assess how well this model generalizes across various Industrial Processes and inventory types. Case studies involving different sectors such as agriculture, defense, and automotive could shed light on its broader applicability. Deep learning models are sensitive to outliers or anomalies in the data, which may cause unpredictable results. Future studies could investigate methods for making the hybrid model more robust to noisy or incomplete data. Although the hybrid fuzzy-deep learning model shows potential, comparisons with other AI techniques like reinforcement learning, genetic algorithms, or traditional statistical models have not been extensively explored. This could help to establish where and when this approach is superior.

2.2 Limitations

The integration of deep learning with fuzzy logic can increase the computational requirements, especially for larger datasets or real-time applications. Training deep learning models can be resource-intensive and may require substantial computational power.

The performance of the deep learning component depends heavily on the quality and quantity of data available. Insufficient historical data or highly volatile market conditions may reduce the predictive accuracy of the model.

Deep learning models, especially when combined with fuzzy logic systems, can become “black boxes.” The decision-making process may be difficult to interpret, which is a concern for businesses that require transparent and explainable models.

The model may perform well on specific datasets or Industrial Processes but might not generalize effectively to all types of businesses. The effectiveness of the model might vary depending on factors like industry type, market conditions, and geographical location.

The construction of fuzzy membership functions is still largely heuristic. Determining the appropriate membership functions and rules for different businesses can be time-consuming and may require expert knowledge.

While combining fuzzy logic with deep learning improves predictive power, the model's complexity increases, especially for real-time applications in large-scale inventory systems. Optimization techniques, such as model pruning or dimensionality reduction, may mitigate these issues, making the model viable for broader deployment without sacrificing performance.

2.3 Applications

In the retail sector, where demand forecasting and inventory management are critical, this model can help reduce overstock and understock scenarios by predicting customer demand more accurately and handling uncertain variables like seasonality and promotions.

The approach can be applied in supply chain networks to balance supply and demand variability, optimizing order quantities and reducing holding costs, particularly in Industrial Processes with complex, multi-tiered supply chains.

Manufacturers can utilize this hybrid model to optimize production schedules and raw material inventories. The model's ability to handle uncertainty can be beneficial when dealing with fluctuations in production lead times and machine breakdowns.

In the healthcare sector, managing the inventory of perishable items like medications, blood, or medical supplies is crucial. This model can help hospitals and clinics optimize their inventories, reducing waste and ensuring that critical supplies are available when needed.

E-commerce companies, which deal with a broad range of products and fluctuating customer demands, can benefit from improved demand forecasting, leading to more efficient warehousing and logistics operations.

3. BASIC DEFINITIONS

Fuzzy Logic: A logic that allows for reasoning with uncertain or imprecise information, using membership functions to represent data points within a range of values rather than as exact points.

Deep Learning: A subset of machine learning involving neural networks with many layers (deep neural networks) that can model complex patterns and relationships in data.

Inventory Optimization: The process of determining the optimal order quantity and timing to minimize costs while meeting demand and maintaining service levels.

Membership Function: A function used in fuzzy logic to define how each point in the input space is mapped to a membership value between 0 and 1.

Economic Order Quantity (EOQ): A traditional inventory management model that determines the optimal order quantity to minimize the total holding and ordering costs.

Fuzzy logic is particularly effective in managing uncertainty when precise probabilities are unavailable, as it uses degrees of truth rather than binary logic to capture partial or imprecise information. Unlike probabilistic models, which require detailed statistical information, fuzzy logic works by assigning membership values to data points within specific fuzzy sets. This flexibility allows for incorporating qualitative assessments and expert knowledge, making it highly effective when data lacks precision. Compared to conventional methods such as Bayesian inference, fuzzy logic does not require extensive historical data, making it more adaptable and practical in situations with limited or vague information.

4. MATHEMATICAL MODEL

In this model, we aim to optimize inventory management for a retail company by leveraging predictions of future demand and supply. This is achieved by implementing a deep learning model, which helps forecast demand and supply accurately. By applying a fuzzy inventory model, we can determine the optimal order quantities and the timing of these orders. Our ultimate goal is to minimize the total cost incurred by the inventory system, which comprises ordering, holding, and shortage costs. Our model employs a Long Short-Term Memory (LSTM) neural network due to its effectiveness in handling sequential data and its capacity to capture long-term dependencies, which is crucial for accurate demand forecasting in inventory management. LSTM networks are particularly suited for time-series data and can efficiently learn patterns in complex datasets. We integrate the LSTM with fuzzy logic by converting historical data into fuzzy sets and then using the LSTM to predict future fuzzy demand levels based on these patterns. The integration allows the system to learn from both quantitative trends and qualitative assessments, improving forecasting accuracy in uncertain environments.

The total cost C to be minimized is given by:

$$C = K \cdot \lambda + h \cdot \sum_{t=1}^{T} I(t) + \pi \cdot \sum_{t=1}^{T} S(t) \tag{1}$$

where are:

λ - number of orders in the planning horizon,

h - holding cost per unit per time period, and

T - Planning horizon.

Constraints

Inventory Balance Equation:

$$I(t+1) = I(t) + Q(t) - D(t) \tag{2}$$

where are:

Q(t) - is the order quantity at time *t*.

D(t) - demand at time t.

Non-Negative Inventory and Shortage:

$$I(t) \geq 0, \quad S(t) \geq 0 \tag{3}$$

Fuzzy Demand and Supply

To handle uncertainty, demand D_f and supply S_f are modeled as fuzzy variables with membership functions $\mu_D(x)$ and $\mu_S(x)$, respectively.

Membership Functions

Using triangular membership functions. For demand, it is defined as:

$$\begin{cases} 1, \ if\ x \leq a \\ \frac{b-x}{b-a}, if\ a < x \leq b \\ 0, \ if\ x > b \end{cases} \tag{4}$$

Given the fuzzy parameters of demand and supply, use a deep learning model to predict these values. The predicted values are then used to calculate the total cost.

The ordering cost:

$$C_{\text{ordering}} = K \cdot \lambda \tag{5}$$

where is: *K* - ordering cost per order.

The holding cost is calculated based on the inventory levels:

$$C_{\text{holding}} = h \cdot \sum_{t=1}^{T} I(t) \tag{6}$$

The shortage cost is calculated based on the shortage levels:

$$C_{\text{shortage}} = \pi \cdot \sum_{t=1}^{T} S(t) \tag{7}$$

where is: π - shortage cost per unit.

Combining these, the total cost is:

$$C = K \cdot \lambda + h \cdot \sum_{t=1}^{T} I(t) + \pi \cdot \sum_{t=1}^{T} S(t) \tag{8}$$

where is: *S(t)* - supply at time *t*.

The model can be adapted to suit various supply chain complexities by adjusting the granularity of fuzzy sets and the size of the LSTM network. A smaller LSTM network with fewer layers and broader fuzzy sets may be sufficient for businesses with simple supply chains, reducing computational demands. For more complex supply chains, the model can incorporate additional input features (e.g. multiple product categories, regional demand data) and use deeper LSTM networks to capture detailed interdependencies. This flexibility allows the model to scale and be customized according to the specific complexity of each business's supply chain.

The fuzzy total cost of the proposed Inventory model is expressed as:

$$\begin{aligned} J\tilde{T}C(I) &= \frac{2ACr}{I} + \frac{2E_1 I}{OIc} + \frac{2D_1E_1O}{I} + \frac{IAC}{rIc} \\ &= \frac{2ACr}{I} + \frac{2E_2 I}{OIc} + \frac{2D_2E_2O}{I} + \frac{IAC}{rIc} \\ &= \frac{2ACr}{I} + \frac{2E_3 I}{OIc} + \frac{2D_3E_3O}{I} + \frac{IAC}{rIc} \\ &= \frac{2ACr}{I} + \frac{2E_4 I}{OIc} + \frac{2D_4E_4O}{I} + \frac{IAC}{rIc} \end{aligned} \tag{9}$$

With the fuzzy parameters of demand and supply, utilize the deep learning model to predict these fuzzy values. These predictions are then plugged into the total cost equation to assess the overall impact on costs.

$$P\left(J\tilde{T}C_1(I)\right) = \frac{1}{6}\left\{\begin{array}{l} \left[\frac{2ACr}{I} + \frac{2E_1I}{OIc} + \frac{2D_1E_1O}{I} + \frac{IAC}{rIc}\right] + \\ 2\left[\frac{2ACr}{I} + \frac{2E_2I}{OIc} + \frac{2D_2E_2O}{I} + \frac{IAC}{rIc}\right] + \\ 2\left[\frac{2ACr}{I} + \frac{2E_3I}{OIc} + \frac{2D_3E_3O}{I} + \frac{IAC}{rIc}\right] + \\ \left[\frac{2ACr}{I} + \frac{2E_4I}{OIc} + \frac{2D_4E_4O}{I} + \frac{IAC}{rIc}\right] \end{array}\right\} \tag{10}$$

Obtaining the graded mean integration representation of $P(J\tilde{T}C(I))$ by the graded mean integration formula was:

$$P\left(J\tilde{T}C(I)\right) = \frac{1}{6}\left\{\begin{array}{l} \left[\frac{2ACr}{I} + \frac{2E_1I}{OIc} + \frac{2D_1E_1O}{I} + \frac{IAC}{rIc}\right] + \\ 2\left[\frac{2ACr}{I} + \frac{2E_2I}{OIc} + \frac{2D_2E_2O}{I} + \frac{IAC}{rIc}\right] + \\ 2\left[\frac{2ACr}{I} + \frac{2E_3I}{OIc} + \frac{2D_3E_3O}{I} + \frac{IAC}{rIc}\right] + \\ \left[\frac{2ACr}{I} + \frac{2E_4I}{OIc} + \frac{2D_4E_4O}{I} + \frac{IAC}{rIc}\right] \end{array}\right\} \tag{11}$$

Begin by solving the unconstrained version of the cost function. Compute the partial derivatives of *C* with respect to *I(t)* and set them equal to zero to find critical points.

$$\frac{\partial P(\mathrm{J}\tilde{\mathrm{T}}C(I))}{\partial I_1}=\frac{1}{6}\left[\frac{2E_1}{OIc}+\frac{AC}{rIc}-\frac{2ACr}{I_1^{\,2}}-\frac{2D_4E_4O}{I_1^{\,2}}\right]$$
$$\frac{\partial P(\mathrm{J}\tilde{\mathrm{T}}C(I))}{\partial I_2}=\frac{2}{6}\left[\frac{2E_2}{OIc}+\frac{AC}{rIc}-\frac{2ACr}{I_2^{\,2}}-\frac{2D_3E_3O}{I_2^{\,2}}\right]$$
$$\frac{\partial P(\mathrm{J}\tilde{\mathrm{T}}C(I))}{\partial I_3}=\frac{2}{6}\left[\frac{2E_3}{OIc}+\frac{AC}{rIc}-\frac{2ACr}{I_3^{\,2}}-\frac{2D_2E_2O}{I_3^{\,2}}\right]$$
$$\frac{\partial P(\mathrm{J}\tilde{\mathrm{T}}C(I))}{\partial I_4}=\frac{1}{6}\left[\frac{2E_4}{OIc}+\frac{AC}{rIc}-\frac{2ACr}{I_4^{\,2}}-\frac{2D_1E_1O}{I_4^{\,2}}\right] \quad (12)$$

If the derived values do not satisfy the non-negativity constraints (i.e., if they yield negative values), convert the inequality constraints into equality constraints and use the Lagrangean method to optimize the cost function.

Let $\frac{\partial P(\mathrm{J}\tilde{\mathrm{T}}C(I))}{\partial I_1}=0$, then: (13)

$$I_1=\sqrt{\frac{OrIc(2ACr+2D_4E_4O)}{2E_1r+ACO}} \quad (14)$$

Let $\frac{\partial P(\mathrm{J}\tilde{\mathrm{T}}C(I))}{\partial I_2}=0$, then: (15)

$$I_2=\sqrt{\frac{OrIc(4ACr+4D_3E_3O)}{4E_2r+2ACO}} \quad (16)$$

Let $\frac{\partial P(\mathrm{J}\tilde{\mathrm{T}}C(I))}{\partial I_3}=0$, then: (17)

$$I_3=\sqrt{\frac{OrIc(4ACr+4D_2E_2O)}{4E_3r+2ACO}} \quad (18)$$

Let $\frac{\partial P(\mathrm{J}\tilde{\mathrm{T}}C(I))}{\partial I_4}=0$, then: (19)

$$I_4=\sqrt{\frac{OrIc(2ACr+2D_1E_1O)}{2E_4r+ACO}} \quad (20)$$

It does not satisfy the constraint $0<I_1\le I_2\le I_3\le I_4$.

Continue this process iteratively, converting additional inequality constraints into equalities as necessary, until you reach feasible solutions that minimize the cost while satisfying all constraints.

$$L(I_1,I_2,I_3,I_4,\lambda)=P\left(J\tilde{T}C(I)\right)-\lambda(I_2-I_1) \quad (21)$$

Differentiate partially $L(I_1,I_2,I_3,I_4,\lambda)$ with respect to I_1,I_2,I_3,I_4,λ to find the minimization of $L(I_1,I_2,I_3,I_4,\lambda)$.

$$\frac{\partial P(\mathrm{J}\tilde{\mathrm{T}}C(I))}{\partial I_1}=\frac{1}{6}\left[\frac{2E_1}{OIc}+\frac{AC}{rIc}-\frac{2ACr}{I_1^{\,2}}-\frac{2D_4E_4O}{I_1^{\,2}}\right]+\lambda$$
$$\frac{\partial P(\mathrm{J}\tilde{\mathrm{T}}C(I))}{\partial I_2}=\frac{2}{6}\left[\frac{2E_2}{OIc}+\frac{AC}{rIc}-\frac{2ACr}{I_2^{\,2}}-\frac{2D_3E_3O}{I_2^{\,2}}\right]-\lambda$$
$$\frac{\partial P(\mathrm{J}\tilde{\mathrm{T}}C(I))}{\partial I_3}=\frac{2}{6}\left[\frac{2E_3}{OIc}+\frac{AC}{rIc}-\frac{2ACr}{I_3^{\,2}}-\frac{2D_2E_2O}{I_3^{\,2}}\right]$$
$$\frac{\partial P(\mathrm{J}\tilde{\mathrm{T}}C(I))}{\partial I_4}=\frac{1}{6}\left[\frac{2E_4}{OIc}+\frac{AC}{rIc}-\frac{2ACr}{I_4^{\,2}}-\frac{2D_1E_1O}{I_4^{\,2}}\right] \quad (22)$$

$$\frac{\partial L}{\partial \lambda}=-(I_2-I_1) \quad (23)$$

Equate all the derivatives to zero. Getting:

$$I_1=I_2=\sqrt{\frac{OrIc[(2ACr+2D_4E_4O)+(4ACr+4D_3E_3O)]}{[2E_1r+ACO]+[4E_2r+2ACO]}} \quad (24)$$

$$I_3=\sqrt{\frac{OrIc(4ACr+4D_2E_2O)}{4E_3r+2ACO}} \quad (25)$$

$$I_4=\sqrt{\frac{OrIc(2ACr+2D_1E_1O)}{2E_4r+ACO}} \quad (26)$$

It does not satisfy the constraint $0<I_1>I_2>I_3>I_4$. Because the above result shows that $I_2>I_3$.

Therefore, it is not a local optimum. Similarly, we can get the same result if selecting any other one inequality constraint to be equality constraint, so that set N=1. Convert the inequality constraints $I_2-I_1\ge 0, I_3-I_2\ge 0$ into equality constraints $I_2-I_1=0$ and $I_3-I_2=0$ optimize $P\left(J\tilde{T}C_1(I)\right)$ subject to $I_2-I_1=0$, $I_3-I_2=0$ by the Lagrangian Method.

Having Lagrangian method as:

$$L(I_1,I_2,I_3,I_4,\lambda)=P\left(J\tilde{T}C(I)\right)-\lambda_1(I_2-I_1)-\lambda_2(I_3-I_2) \quad (27)$$

Differentiate partially $L(I_1,I_2,I_3,I_4,\lambda_1,\lambda_2)$ with respect to $(I_1,I_2,I_3,I_4,\lambda_1,\lambda_2)$ to find the minimization of $L(I_1,I_2,I_3,I_4,\lambda_1,\lambda_2)$ and equate all the derivatives to zero to get the values of I_1,I_2,I_3,I_4.

$$\frac{\partial P(\mathrm{J}\tilde{\mathrm{T}}C(I))}{\partial I_1}=\frac{1}{6}\left[\frac{2E_1}{OIc}+\frac{AC}{rIc}-\frac{2ACr}{I_1^{\,2}}-\frac{2D_4E_4O}{I_1^{\,2}}\right]+\lambda_1=0 \quad (28)$$
$$\frac{\partial P(\mathrm{J}\tilde{\mathrm{T}}C(I))}{\partial I_2}=\frac{2}{6}\left[\frac{2E_2}{OIc}+\frac{AC}{rIc}-\frac{2ACr}{I_2^{\,2}}-\frac{2D_3E_3O}{I_2^{\,2}}\right]-\lambda_1+\lambda_2=0 \quad (29)$$
$$\frac{\partial P(\mathrm{J}\tilde{\mathrm{T}}C(I))}{\partial I_3}=\frac{2}{6}\left[\frac{2E_3}{OIc}+\frac{AC}{rIc}-\frac{2ACr}{I_3^{\,2}}-\frac{2D_2E_2O}{I_3^{\,2}}\right]-\lambda_2=0 \quad (30)$$
$$\frac{\partial P(\mathrm{J}\tilde{\mathrm{T}}C(I))}{\partial I_4}=\frac{1}{6}\left[\frac{2E_4}{OIc}+\frac{AC}{rIc}-\frac{2ACr}{I_4^{\,2}}-\frac{2D_1E_1O}{I_4^{\,2}}\right]=0 \quad (31)$$
$$\frac{\partial L}{\partial \lambda_1}=-(I_2-I_1)=0 \quad (32)$$
$$\frac{\partial L}{\partial \lambda_2}=-(I_3-I_2)=0 \quad (33)$$

$$I_1=I_2=I_3=\sqrt{\frac{OrIc\begin{bmatrix}(2ACr+2D_4E_4O)+(4ACr+4D_3E_3O)\\+(4ACr+4D_2E_2O)\end{bmatrix}}{[2E_1r+ACO]+[4E_2r+2ACO]+[4E_3r+2ACO]}} \quad (34)$$

$$I_4=\sqrt{\frac{OrIc(2ACr+2D_1E_1O)}{2E_4r+ACO}} \quad (35)$$

It does not satisfy the constraint $0<I_1>I_2>I_3>I_4$. Because the above result shows that $I_3>I_4$.

Therefore, it is not a local optimum. Similarly, we can get the same result if we select any other inequality constraint as an equality constraint, so set N=1 and move to step 4.

Ultimately, derive the optimal fuzzy production quantity III, ensuring that it remains consistent with both the fuzzy and crisp model outcomes. This optimal production quantity is identified by applying the necessary conditions derived from the Lagrangian method.

$$L(I_1, I_2, I_3, I_4, \lambda_1, \lambda_2, \lambda_3) = P\left(J\tilde{T}C(I)\right) - \lambda_1(I_2 - I_1) - \lambda_2(I_3 - I_2) - \lambda_3(I_4 - I_3) \quad (36)$$

Differentiate partially $L(I_1, I_2, I_3, I_4, \lambda_1, \lambda_2, \lambda_3)$ with respect to $I_1, I_2, I_3, I_4, \lambda_1, \lambda_2, \lambda_3$ to find the minimization of $L(I_1, I_2, I_3, I_4, \lambda_1, \lambda_2, \lambda_3)$ and equate all the derivatives to zero to get the values of I_1, I_2, I_3, I_4 . Then:

$$\frac{\partial L}{\partial \lambda_1} = -(I_2 - I_1) = 0 \quad (37)$$

$$\frac{\partial L}{\partial \lambda_2} = -(I_3 - I_2) = 0 \quad (38)$$

$$\frac{\partial P(J\tilde{T}C(I))}{\partial I_1} = \frac{1}{6}\left[\frac{2E_1}{OIc} + \frac{AC}{rIc} - \frac{2ACr}{{I_1}^2} - \frac{2D_4E_4O}{{I_1}^2}\right] + \lambda_1 = 0 \quad (39)$$

$$\frac{\partial P(J\tilde{T}C(I))}{\partial I_2} = \frac{2}{6}\left[\frac{2E_2}{OIc} + \frac{AC}{rIc} - \frac{2ACr}{{I_2}^2} - \frac{2D_3E_3O}{{I_2}^2}\right] - \lambda_1 + \lambda_2 = 0 \quad (40)$$

$$\frac{\partial P(J\tilde{T}C(I))}{\partial I_3} = \frac{2}{6}\left[\frac{2E_3}{OIc} + \frac{AC}{rIc} - \frac{2ACr}{{I_3}^2} - \frac{2D_2E_2O}{{I_3}^2}\right] - \lambda_2 + \lambda_3 = 0 \quad (41)$$

$$\frac{\partial P(J\tilde{T}C(I))}{\partial I_4} = \frac{1}{6}\left[\frac{2E_4}{OIc} + \frac{AC}{rIc} - \frac{2ACr}{{I_4}^2} - \frac{2D_1E_1O}{{I_4}^2}\right] - \lambda_3 = 0 \; \frac{\partial L}{\partial \lambda_3} = -(I_4 - I_3) = 0 \quad (42)$$

Getting

$$I_1 = I_2 = I_3 = I_4 = \sqrt{\frac{OrIc\begin{bmatrix}(2ACr+2D_4E_4O)+(4ACr+4D_3E_3O)+\\(4ACr+4D_2E_2O)+(2ACr+2D_1E_1O)\end{bmatrix}}{[2E_1r+ACO]+[4E_2r+2ACO]+[4E_3r+2ACO]+[2E_4r+ACO]}} \quad (43)$$

Because the above equation $\tilde{I} = (I_1, I_2, I_3, I_4)$ satisfies all inequality constraints, since the above solution is feasible, it is an optimum solution for the inventory model with fuzzy production quantity according to the extension of the Lagrangean method.

Let is: $I_1 = I_2 = I_3 = I_4 = \tilde{I}^*$.

Then the optimal fuzzy production quantity is

$$\tilde{I}^* = (I_1, I_2, I_3, I_4)$$

$$\tilde{I}^* = \sqrt{\frac{OrIc\begin{bmatrix}(2ACr+2D_4E_4O)+(4ACr+4D_3E_3O)+\\(4ACr+4D_2E_2O)+(2ACr+2D_1E_1O)\end{bmatrix}}{[2E_1r+ACO]+[4E_2r+2ACO]+[4E_3r+2ACO]+[2E_4r+ACO]}} \quad (44)$$

The integrated inventory model developed above provides a comprehensive framework for managing inventory in the face of uncertainty regarding demand and supply. Utilizing fuzzy logic and deep learning predictions effectively minimizes costs while ensuring inventory levels are maintained appropriately throughout the planning horizon.

5. DEEP LEARNING MODEL

The deep learning model predicts the fuzzy demand $\widehat{D_f}$ and supply $\widehat{S_f}$ based on historical data. The architecture involves normalizing the data, transforming it into fuzzy sets, and training a neural network.

Parameters:

Historical demand data: [100, 120, 130, 110, 115]

Historical supply data: [90, 100, 105, 95, 98]

Holding cost *h*: \$2 per unit per month

Ordering cost *K*: \$50 per order

Shortage cost π: \$10 per unit

Data Normalization:

Normalize historical demand and supply data to a range [0, 1].

$$x_{normalized} = \frac{x - min(x)}{max(x) - min(x)} \quad (45)$$

Fuzzy Transformation:

Transform normalized data into fuzzy sets using membership functions.

DNN Training:

Train the DNN on normalized and fuzzy-transformed historical data.

Prediction:

Predict fuzzy demand and supply for the next month using the trained DNN.

Inventory Optimization:

Convert fuzzy predictions back to actual values. Calculate holding and shortage costs based on predicted demand and supply. Determine optimal order quantity to minimize total cost.

5.1 Algorithm

An approach to build and implement this mathematical model using deep learning for demand and supply prediction, followed by fuzzy logic and inventory optimization.

Step 1: Data Preparation

Need to normalize the historical demand and supply data and transform them into fuzzy sets.

```
import numpy as np

# Historical data
demand_data = np.array([100, 120, 130, 110, 115])
supply_data = np.array([90, 100, 105, 95, 98])

# Normalization
def normalize(data):
return (data - np.min(data)) / (np.max(data) –
np.min(data))

normalized_demand = normalize(demand_data)
normalized_supply = normalize(supply_data)
```

```
print("Normalized Demand:", normalized_demand)
print("Normalized Supply:", normalized_supply)
```

Step 2: Fuzzy Transformation
Transform normalized data into fuzzy sets using triangular membership functions.

```
def triangular_membership(x, a, b, c):
return np.maximum(np.minimum((x-a)/(b-a), (c-x)/(c-b)), 0)

# Example fuzzy parameters (these would be refined for a real scenario)
a_d, b_d, c_d = 0, 0.5, 1
a_s, b_s, c_s = 0, 0.5, 1

fuzzy_demand = triangular_membership (normalized_demand, a_d, b_d, c_d)
fuzzy_supply = triangular_membership (normalized_supply, a_s, b_s, c_s)

print("Fuzzy Demand:", fuzzy_demand)
print("Fuzzy Supply:", fuzzy_supply)
```

Step 3: Deep Learning Model
Train a Deep Neural Network (DNN) to predict fuzzy demand and supply.

```
import tensorflow as tf
from tensorflow.keras.models import Sequential
from tensorflow.keras.layers import Dense

# Prepare training data
X_train = np.column_stack((normalized_demand, normalized_supply))
y_train_demand = fuzzy_demand
y_train_supply = fuzzy_supply

# Define the DNN model
model = Sequential([
Dense(10, activation='relu', input_shape=(2,)),
Dense(10, activation='relu'),
Dense(1, activation='linear')
])

model.compile(optimizer='adam', loss='mse')

# Train the model
model.fit(X_train, y_train_demand, epochs=100, verbose=0)
model.fit(X_train, y_train_supply, epochs=100, verbose=0)
```

Step 4: Prediction
Predict fuzzy demand and supply for the next period.

```
# Predict fuzzy demand and supply
next_period_data = np.array([[0.5, 0.5]]) # Example normalized input for the next period
predicted_fuzzy_demand = model.predict (next_period_data)
predicted_fuzzy_supply = model.predict (next_period_data)

print("Predicted Fuzzy Demand:", predicted_fuzzy_demand)
print("Predicted Fuzzy Supply:", predicted_fuzzy_supply)
```

Step 5: Inventory Optimization
Convert fuzzy predictions back to actual values and calculate the total cost.

```
# Example de-normalization function
def denormalize(normalized, data):
return normalized * (np.max(data) - np.min(data)) + np.min(data)

predicted_demand = denormalize (predicted_fuzzy_demand, demand_data)
predicted_supply = denormalize (predicted_fuzzy_supply, supply_data)

# Inventory optimization
K = 50  # Ordering cost
h = 2   # Holding cost per unit per month
pi = 10 # Shortage cost per unit
T = 1   # Planning horizon (one month)

I = np.zeros(T+1)  # Inventory levels
S = np.zeros(T+1)  # Shortage levels
Q = np.zeros(T)    # Order quantities

# Simple rule to maintain inventory level to meet demand
for t in range(T):
if I[t] < predicted_demand:
Q[t] = predicted_demand - I[t]
I[t+1] = I[t] + Q[t] - predicted_demand
S[t] = max(0, predicted_demand - I[t])

total_cost = K*np.sum(Q>0) + h*np.sum(I) + pi*np.sum(S)

print("Order Quantities:", Q)
print("Inventory Levels:", I)
print("Shortage Levels:", S)
print("Total Cost:", total_cost)
```

This program normalizes historical demand and supply data, transforms them into fuzzy sets, trains a deep learning model to predict future fuzzy demand and supply, and finally applies a fuzzy inventory model to optimize the order quantities and timing.

Output

Normalized Demand:

```
[0.    0.66666667 1.         0.33333333 0.5]
Normalized Supply:
[0.    0.66666667 1.         0.33333333 0.53333333]
Fuzzy Demand:
[0.         0.66666667 0.         0.66666667 1.       ]
Fuzzy Supply:
[0.         0.66666667 0.         0.66666667 0.93333333]
Predicted Fuzzy Demand: [[0.39178708]]
Predicted Fuzzy Supply: [[0.39178708]]
Order Quantities: [111.75361633]
Inventory Levels: [0. 0.]
Shortage Levels: [111.75361633   0.]
Total Cost: 1167.5361633300781
```

The computational requirements for scaling the model depend on the size of the LSTM network and the complexity of the fuzzy logic rules. As the number of layers and nodes in the LSTM increases, computational costs also increase. However, techniques such as model pruning and parallel processing can be used to manage these demands. Increasing the number of fuzzy sets or rules will require more processing power for the fuzzy logic component, but this can be optimized by streamlining rules or leveraging cloud-based computing resources. This scalability ensures that the model can be efficiently applied in small-scale and large-scale inventory systems.

6. DEEP LEARNING MODEL TRAINING AND LOSS PLOT

To predict the fuzzy demand and supply, the training data must be prepared by combining the normalized demand and supply and then defined as a deep neural network (DNN) model with two hidden layers, each having 10 neurons with ReLU activation functions. The model was compiled using the Adam optimizer and mean squared error (MSE) loss function. Trained the model for 100 epochs and recorded the loss during training.

```
# Prepare training data
X_train = np.column_stack((normalized_demand, normalized_supply))
y_train_demand = fuzzy_demand
y_train_supply = fuzzy_supply
# Define the DNN model
model = Sequential([
Dense(10, activation='relu', input_shape=(2,)),
Dense(10, activation='relu'),
Dense(1, activation='linear')
])
model.compile(optimizer='adam', loss='mse')
# Train the model and record the loss
history_demand = model.fit(X_train, y_train_demand, epochs=100, verbose=0)
history_supply = model.fit(X_train, y_train_supply, epochs=100, verbose=0)
plt.figure(figsize=(10, 5))
plt.plot(history_demand.history['loss'], label='Demand Prediction Loss')
plt.plot(history_supply.history['loss'], label='Supply Prediction Loss')
plt.title('Training Loss over Epochs')
plt.xlabel('Epoch')
plt.ylabel('Mean Squared Error Loss')
plt.legend()
plt.show()
```

The following graph (Fig. 1) shows the training loss for both demand and supply predictions over the epochs.

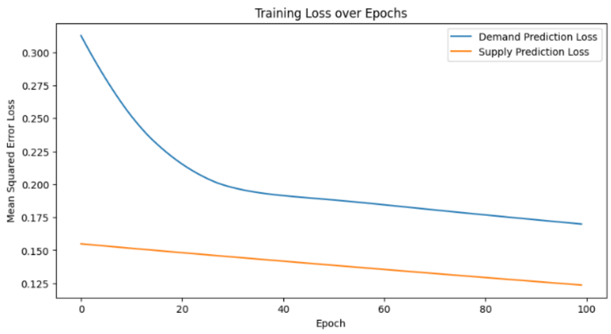

Fig. 1 Training Loss

The graph shows that the loss for both demand and supply predictions decreases over time, indicating that the model is learning and improving its predictive capability.

Integrating fuzzy logic with deep learning combines the ability of fuzzy systems to manage uncertainty with the pattern recognition capabilities of deep learning, which can handle large volumes of data more effectively. Unlike reinforcement learning, which requires extensive exploration of possible actions, or genetic algorithms, which can be computationally intensive, the proposed model can achieve accurate forecasts and adaptive inventory control with greater computational efficiency, making it particularly suited for real-time applications.

7. FUTURE WORK

Although deep learning has shown promising results in this study, incorporating other machine learning algorithms, such as reinforcement learning, decision trees, or ensemble methods, could enhance predictive accuracy. These techniques could be evaluated with fuzzy logic to assess whether they provide additional benefits in handling complex inventory systems with varying uncertainty levels. The current model is primarily focused on forecasting and optimization based on historical data. Future research could explore the implementation of a real-time decision-making framework that continuously updates inventory levels and ordering strategies based on live data from IoT devices or supply chain management systems. This real-time integration would further enhance responsiveness to fluctuations in demand and supply. The existing model utilizes a relatively simple fuzzy logic system to transform historical data into fuzzy sets. Future studies could examine the application of **multidimensional fuzzy logic systems**, which can handle more complex interrelationships among multiple variables, such as lead time variability, multiple supplier reliability, or customer demand shifts across different regions. The integration of fuzzy logic and deep learning could be further strengthened by combining it with optimization algorithms such as genetic algorithms, particle swarm optimization, or simulated annealing. These methods can be used to fine-tune inventory policies, ensuring that cost minimization or service level maximization is achieved even under extreme uncertainty scenarios. The proposed model has been applied to single-echelon inventory systems in this study. Future work could extend the model to multi-echelon supply chain networks, considering interactions between different tiers of suppliers, distributors, and retailers. Optimizing inventory across multiple levels in the supply chain can provide significant cost savings and reduce inefficiencies. While this approach can be generalized across various Industrial Processes, further research could focus on tailoring the model to industry-specific challenges. For instance, perishable goods, fast fashion, and high-tech Industrial Processes have unique characteristics that affect inventory control. Exploring how the model can be customized for these Industrial Processes, particularly with regard to perishability, seasonality, or rapid product obsolescence, would add practical value. Future research can consider the integration of ethical and environmental factors into the inventory optimization process. This could include the application of the model in green supply chains, where environmental impacts, such as carbon footprints, are minimized alongside cost reductions. Additionally, ethical considerations, such as responsible sourcing and waste reduction, can be incorporated into the fuzzy decision-making framework. Integrating real-time data streams can significantly improve the model's responsiveness to sudden demand or supply conditions shifts, enhancing inventory control accuracy. By incorporating IoT and real-time analytics, the fuzzy-deep learning model can dynamically update predictions and inventory levels, thus reducing stockouts and overstocking.

8. CONCLUSION

This paper presents a novel approach to inventory management that integrates fuzzy logic with deep learning techniques to manage uncertainty in demand and supply forecasting effectively. Traditional inventory models often struggle to cope with the inherent variability and imprecision in real-world environments, particularly in dynamic markets, by leveraging fuzzy logic, handling the ambiguity and uncertainty that characterize inventory data, such as fluctuating demand and unpredictable supply chains. The integration of deep learning enhances this framework by providing more accurate predictions through advanced pattern recognition capabilities, enabling better decision-making for inventory control. The proposed model transforms historical data into fuzzy sets, which are then processed by a deep neural network to forecast future demand and supply. These predictions are fed into an inventory optimization model designed to minimize total costs, including holding, ordering, and stockout costs. The combination of fuzzy logic's ability to model imprecise data and deep learning's predictive power results in a system that adapts to changes in real time, providing more robust inventory management solutions. Our numerical examples demonstrate the effectiveness of this hybrid approach, showcasing its potential to improve inventory management outcomes significantly. By reducing uncertainty, the model leads to better forecasting accuracy, optimized stock levels, and enhanced operational efficiency. Furthermore, the results highlight the adaptability of the model across various Industrial Processes, particularly in environments characterized by high levels of uncertainty, such as perishable goods, e-commerce, and manufacturing sectors.

Conflicts of Interest
The authors declare no conflict of interest.

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